

Error Attenuation in Abel Inversion

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A data approximation technique is presented for solving numerically the Abel integral equation. The side-on intensity is developed in a set of even powers. The optimization of the number of terms in the expansion makes it possible to control the smoothing of the data, and thereby strongly reduces the effects of experimental errors. The results are compared with those obtained by other methods, demonstrating the utility of this technique.

1. INTRODUCTION

If the emissivity ϵ of an optically thin cylindrical plasma is assumed to depend only on the radial variable r , the side-on intensity may be written

$$I(x) = 2 \int_0^{(r_0^2 - x^2)^{1/2}} \epsilon(r) dy, \quad y^2 = r^2 - x^2, \quad (1.1)$$

where x and y are Cartesian coordinates, and r_0 is the outer radius of the cylinder (see Fig. 1). By changing the integration variable, this equation becomes

$$I(x) = 2 \int_x^{r_0} \frac{r\epsilon(r)}{(r^2 - x^2)^{1/2}} dr, \quad (1.2)$$

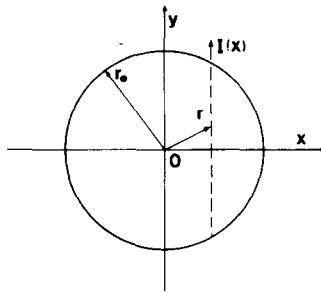


FIG. 1. Geometry of the problem.

and by applying the Abel inversion formula [1], the solution of Eq. (1.1) is given by

$$\varepsilon(r) = \frac{1}{\pi} \left\{ \frac{I(r_0)}{(r_0^2 - r^2)^{1/2}} - \int_r^{r_0} \frac{I'(x)}{(x^2 - r^2)^{1/2}} dx \right\}, \quad (1.3a)$$

where $I'(x)$ denotes the first derivative of I with respect to x . Since the plasma is not irradiated by any external source, $I(r_0) = 0$, and hence Eq. (1.3a) reduces to

$$\varepsilon(r) = -\frac{1}{\pi} \int_r^{r_0} \frac{I'(x)}{(x^2 - r^2)^{1/2}} dx. \quad (1.3b)$$

The measurements give a set of N values of $I(x)$. From these values, the emissivity is to be determined either by solving integral equation (1.1), or by evaluating Eq. (1.3). Different techniques have been proposed to solve this problem. They may be divided into two classes: numerical methods and data approximation techniques.

The numerical methods are based on a common principle: either the side-on intensity or the emissivity is assumed to have a variation given by a particular law over a small interval, namely, the distance between two points of measurements (see, for example, [2-7]). The emissivity is generally obtained by a recurrence formula, the computation starting at $r = r_0$ and ending at $r = 0$. The measurements near the edge being the most difficult, the determination of the emissivity in this region is not easy, and hence, it must be expected that the calculated distribution will involve large uncertainties due to error propagation.

In data approximation techniques, $I(x)$ is developed in a set of basis functions ϕ_i , chosen for some particular properties (see, for example, [8-12]). The number of such functions is either arbitrarily fixed or optimized. This optimization makes it possible to obtain and control the smoothing of the measurements, and thereby reduces the effects of the experimental errors.

2. SUGGESTED METHOD

The experiment to which the method will be applied, dictates the following limitations:

- (i) a reduced number of points (the maximum is 21),
- (ii) unequally spaced points.

It is also desirable to smooth the data in order to eliminate experimental errors.

The side-on intensity is approximated by a polynomial:

$$I(x) = \sum_{i=0}^k \alpha_i x^i, \quad (2.1)$$

where the coefficients α_i are determined by minimizing the errors with respect to the

experimental data, using a least-squares approximation. The system of equations thus obtained is solved by the modified Cholesky method [13].

The emissivity $\varepsilon(r)$ is then given by analytical integration of (1.3) using the following formulae:

$$\int \frac{dx}{s} = \ln |x + s|,$$

$$\int \frac{x^{2m}}{s} dx = \frac{x^{2m-1}}{2m} s + \frac{2m-1}{2m} r^2 \int \frac{x^{2(m-1)}}{s} dx, \quad m > 0; \quad (2.2a)$$

$$\int \frac{x^{2m+1}}{s} = \sum_{j=0}^m \binom{m}{j} \frac{s^{2j+1}}{2j+1} r^{2(m-j)}, \quad m \geq 0; \quad (2.2b)$$

where $s = (x^2 - r^2)^{1/2}$ and $\binom{m}{j}$ represents the binomial coefficients. Since, in the present problem, the emissivity depends only on the radial variable r , $I(x)$ must be an even function, and hence, its development will contain only even powers of x . Thus, only Eq. (2.2b) is needed.

Additional formulae may be obtained from Eqs. (1.2) and (1.3b), using the relation (2.1). If

$$I(x) = \sum_{i=0}^k \alpha_{2i} x^{2i}, \quad \text{with } |x| \leq r_0 = 1,$$

then

$$I'(1) = \sum_{i=0}^k 2i \alpha_{2i}; \quad (2.3a)$$

$$\varepsilon(r=0) = -\frac{1}{\pi} \sum_{i=0}^k \frac{2i}{2i-1} \alpha_{2i}; \quad (2.3b)$$

$$\varepsilon(1-\delta) \simeq -(1/\pi)(2\delta)^{1/2} I'(1), \quad \delta \ll 1. \quad (2.3c)$$

3. OPTIMIZATION OF THE DEGREE OF THE BASIS FUNCTIONS

3.1. Criterion of the Least Square Approximation

The measurements give N couples (x_i, I_i) and the polynomial expansion gives N couples $(x_i, I(x_i))$. A measure of the goodness of the resulting approximation [14] is

$$S = (\sigma^2 / (N - m))^{1/2}, \quad (3.1)$$

where

$$\sigma^2 = \sum_{i=1}^N (I_i - I(x_i))^2;$$

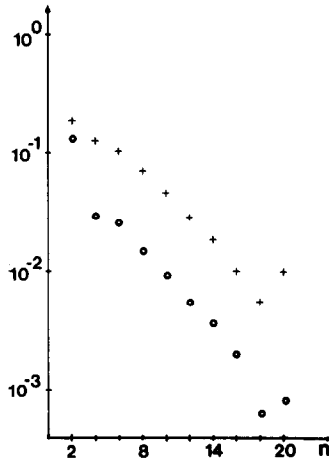


FIG. 2. Test Function 3; ($N = 23$, 3 decimal places) (○) S ; (+) σ_{rad} .

m is the number of basis functions in the development; and $m = (n/2) + 1$, where n is the maximum degree of development.

Figure 2 shows the variation of S as a function of n , the maximum degree of the basis functions, on a semilogarithmic diagram (the execution of the tests is explained in Section 5). The optimal degree is determined by the minimum of S [15], as the comparison between S and σ_{rad} shows. The latter allows the determination of the actual optimum degree

$$\sigma_{rad}^2 = \frac{1}{N} \sum_{i=1}^N (\epsilon_{exact}(r_i) - \epsilon_{calculated}(r_i))^2. \tag{3.2}$$

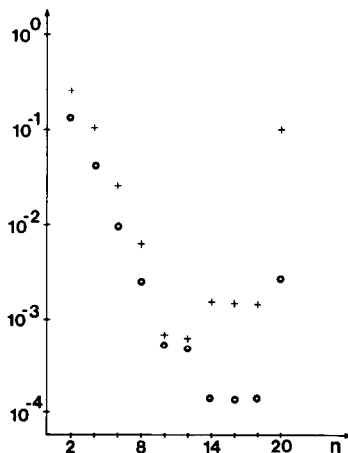


FIG. 3. Test Function 1; ($a = 0.5$), ($N = 23$, 3 decimal places), (○) S ; (+) σ_{rad} .

If the measurements were accurate, an increase in the degree n would lead to a decrease in σ^2 and therefore a decrease in S . After the minimum value of σ^2 has been reached (minimum error of polynomial approximation), any new increase in n would produce an increase in S . When experimental errors are considered, the same criterion makes it possible to control the smoothing of the data. The case shown in Fig. 2 is in fact an ideal one, the degree being unequivocally defined. The choice is not always so obvious. This problem is illustrated in Fig. 3: the optimal degree is given by the beginning of the level $n = 12$, and not by the minimum of S .

3.2. Effects of the Peripheral Intensity

As mentioned in the introduction, $I(r_0) = 0$, and therefore, the choice of the optimal degree n_{opt} should be subject to this constraint. Figure 4 shows the result of taking into account the value of $I(r_0)$ obtained for the different maximum degrees. The criterion based on S gives $n_{\text{opt}} = 16$ while the behaviour of $I(r_0)$ dictates clearly that $n_{\text{opt}} = 10$. In addition, relation (2.3c) suggests that a criterion for the choice of the optimum degree might also be $I'(r_0) < 0$.

Three complementary criteria are thus available to help us choose the optimum degree. It must be noted that the choice, among the selected degrees, should be the lowest possible value. In fact, too high a degree reduces the smoothing, and hence, generates undesirable oscillations.

4. PARTICULAR RADIAL DISTRIBUTION

4.1. Discontinuous Emissivity

The test functions used by other authors ([16, 17], for example) are generally chosen so that $\varepsilon(r_0) = 0$ or $\varepsilon(r_0) \ll \varepsilon_{\text{max}}$, with the exception of those taken from [18].

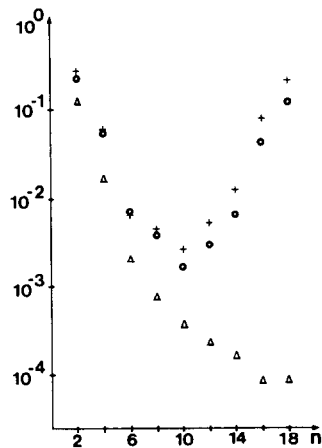


FIG. 4. Test function 5; ($N = 21$, 3 decimal places) (Δ) S ; (+) $I(r_0)$; (\circ) σ_{rad} .

TABLE I

Number of points	$N = 21$	$N = 23$
n_{opt}	16.0	20.0
$I(r_0)$	0.164	0.0
$I'(r_0)$	-13.8	-23.1

It seemed desirable, however, to study also the behaviour of the written program for an emissivity with $\varepsilon(r_0) \neq 0$. If

$$\varepsilon(r) = \varepsilon_1(r) + A(1 - U(r - r_0)),$$

where

$$\varepsilon_1(r_0) = 0 \tag{4.1}$$

and $U(r - r_0)$ is a step function at $r = r_0$, then

$$I(x) = I_1(x) + 2A(r_0^2 - x^2)^{1/2}, \tag{4.2}$$

and

$$I(r_0) = 0,$$

$$I'(r_0) = -\infty.$$

The behaviour of $I'(r_0)$ may suggest the existence of a nonzero value of the emissivity at $r = r_0$.

The chosen emissivity and intensity distributions are

$$\varepsilon(r) = 1,$$

$$I(x) = 2(r_0^2 - x^2)^{1/2}.$$

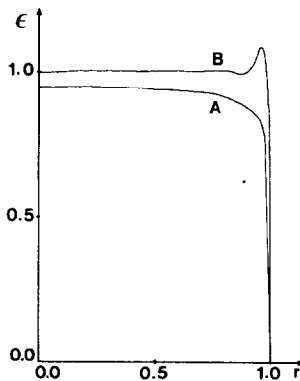


FIG. 5. Test Function 2; $\varepsilon = \varepsilon(r)$; (A) $N = 21$; (B) $N = 23$.

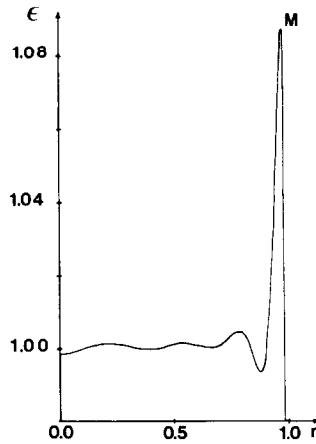


FIG. 6. Test Function 2; ($N = 23$, 3 decimal places), $\varepsilon = \varepsilon(r)$.

Twenty-one data points are symmetrically located around $x = 0$, with $|x| = r_0$ omitted. In a second test, the points $|x| = r_0$ are included. The results are shown in Table I, while Fig. 5 describes the distributions.

By forcing $I(\pm r_0)$ to be zero, the calculated distribution is actually improved. The value of $I'(r_0)$, already large, has increased in the second test, so that it is consistent to have a nonzero value for $\varepsilon(r_0)$. It should be noted that the method does not give this value, which must therefore be determined by graphical extrapolation.

4.2. Oscillations and Optimum Degree

The radial distribution of $\varepsilon(r)$ given by the test with 23 points exhibits nonnegligible oscillations for $r \geq 0.9r_0$. This phenomenon appears in a similar way in the Fourier series representation of a discontinuous function (Gibbs phenomenon). The oscillations for $r \leq 0.9r_0$ are inherent in the method [10]. They can not, however, be considered as real variations in the emissivity, since their values are below one per cent (see Fig. 6). The Gibbs phenomenon gives a justification of the criterion for choosing the optimum degree. The coordinates (r, ε) of the point where the overshoot is a maximum (point M in Fig. 6) are shown graphically for various maximum degrees of the polynomials (Fig. 7). The degree $n = 20$ behaves as a limit of stability with regard to the oscillations. This fact seems to confirm the validity of the criterion (see Table I).

5. COMPARISON BETWEEN DIFFERENT METHODS

To check the FORTRAN code, tests have been carried out on a CDC 6600, with radial distributions for which the side-on intensity is known analytically (see formulae (2.2) and the Appendix).

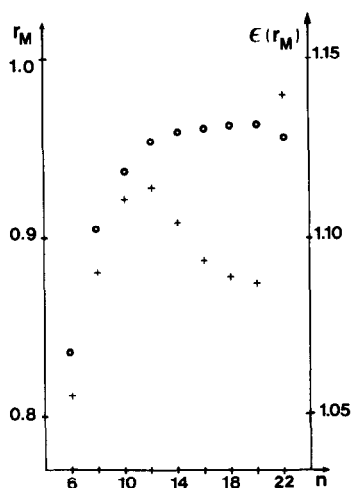


FIG. 7. Test Function 2; ($N = 23$, 3 decimal places), coordinates (r, ϵ) of the point M as a function of n , (O) r_M ; (+) $\epsilon(r_M)$.

For data, N points $(x_i, I(x_i))$ are chosen. $I(x_i)$ is defined with limited precision, subordinate to the number of decimal places. It should be noted that such data are not strictly random. The standard deviations, calculated by (3.2) are compared with results presented by [10, 16, 17] (see Tables II, III).

TABLE II
Test Function 5: $\epsilon(r) = 1 - 3r^2 + 2r^3$ ($r_0 = 1$)

Decimal places	Number of points	Nestor, Olsen [3]	Landenburg <i>et al.</i> [2]	Frie [4]	Cremers, Birkebak [10]	Glasser <i>et al.</i> [16]	This work
exact	11	0.0125	0.0033	0.0024	0.180	0.004	0.00021
	21	0.0046	0.0008	0.0006	0.004	0.00007	0.00020
2	11	0.0139	0.0104	0.0839	0.1770	0.0035	0.0029
	21	0.0113	0.0144	0.0125	0.0067	0.0057	0.0037

TABLE III

Test Functions	Decimal Places ($N = 21$)	Frie [4]	Edels <i>et al.</i> [5]	Barr [9]	Bockasten [8]	Glasser <i>et al.</i> [16]	This work
1	3	0.017	0.010	0.006	0.031	0.0007	0.0006
($a = 0.5$)	1	0.242	0.528	0.025	0.100	0.020	0.022
4	3	0.007	0.006	0.003	0.005	0.0006	0.0005
($a = 0.5$)	3	0.031	0.028	0.013	0.036	—	0.006

As expected, the data approximation techniques give better results than the numerical method. Though the accuracy of all methods decreases with increasing data scattering, the comparative accuracy of the data approximation techniques improves since they permit a smooth approximation to be obtained from the data. In spite of the problem described in Section 3.1, the optimum degree may be objectively chosen. When the three criteria ($S, I(r_0), I'(r_0)$) are applied together, the possibilities of the choice are reduced to one or two values which give almost identical results. This property seems to be an advantage, compared with the interactive control of the smoothing in [16]. It must be pointed out that the single criterion based on S [12] may lead to negative emissivity which is physically unacceptable. This criterion must therefore be completed by the two others.

6. CONCLUSION

The proposed technique has the ability to work with unequally spaced and nonnumerous data which contain experimental errors.

The main contribution of this method which is easily adapted to computer calculation, is the choice of the optimum degree of the polynomial expansion of the side-on intensity. Although this degree may not be unequivocally determined, the developed criteria indicate good reliability.

Results obtained either with theoretical distributions (see Tables II, III) or with experimental data [11, 19] confirm the validity of the technique presented in this paper.

APPENDIX: TEST FUNCTIONS

$$\text{T.F. 1.} \quad \varepsilon(r) = \exp(-r^2/a^2),$$

$$I(x) = a\pi^{1/2} \exp(-x^2/a^2) \operatorname{erf}((r_0^2 - x^2)^{1/2}/a).$$

$$\text{T.F. 2.} \quad \varepsilon(r) = 1,$$

$$I(x) = 2(r_0^2 - x^2)^{1/2}.$$

$$\text{T.F. 3.} \quad \varepsilon(r) = \frac{3}{4} + 12r^2 - 32r^3, \quad \text{if } 0.0 \leq r \leq 0.25,$$

$$= \frac{16}{27}(1 + 6r - 15r^2 + 8r^3), \quad \text{if } 0.25 \leq r \leq r_0 = 1.$$

$I(x)$ obtained by integration of (1.2).

$$\text{T.F. 4.} \quad \varepsilon(r) = r^2 \exp(-r^2/a^2)$$

$$I(x) = a\pi^{1/2} \exp(-x^2/a^2) \operatorname{erf}((r_0^2 - x^2)^{1/2}/a)(x^2 + a^2/2) \\ - a^2 \exp(-r_0^2/a^2)(r_0^2 - x^2)^{1/2}$$

$$\begin{aligned} \text{T.F. 5.} \quad \varepsilon(r) &= 1 - 3r^2 + 2r^3 \quad (r_0 = 1) \\ I(x) &= (1 - x^2)^{1/2}(1 - 5/2x^2) + 3/2x^4 \ln |(1 + (1 - x^2)^{1/2})/x|. \end{aligned}$$

Except when otherwise stated, the tests are performed with $r_0 = 1$.

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